Mathematical Investigations and Concepts for Algebraic Reasoning – A Prior-to-Algebra High school Course

This course is designed to address the needs of students who begin high school without the necessary mathematical background to enroll in an Algebra 1 or equivalent course. The focus of the course will be on the conceptual development of the mathematical content and skills necessary to successfully complete an Algebra 1 course and courses beyond Algebra 1. Much of the course is devoted to addressing major misconceptions that have contributed to students' misunderstandings with mathematics. The course will be integrated in nature (i.e., integrating concepts from number and operations, geometry and measurement, functions and algebra, and data, statistics, and probability) and follow the guidelines set forth in *Principles and Standards for School Mathematics* (NCTM, 2000). The course is student-centered and focuses on activity-based instruction that integrates technology and emphasizes the conceptual understanding of the mathematical concepts studied. Multiple assessment tools will be used to measure the course competencies, including on-going formative assessments, diagnostic tasks, portfolios, quizzes, exams, and projects/investigations. The instructor should explore the resources listed in Appendix A and pay particular attention to those resources that highlight research related to students' misconceptions around topics in this course. The recommended Distribution of Emphasis for the particular content strands can be found in Appendix B. This emphasis does not translate directly to instructional time (e.g., some topics in statistics may take longer to develop than topics in number and operations). However, the emphasis represents the percent of score points in each of the strands that would be recommended for a final project, final exam, capstone project, or other means of final assessment. Additionally, Appendix C contains some of the important Habits of the Mind that are referenced in the course process skills.

Content Strands:

Number and Operations
Geometry and Measurement
Functions and Algebra
Data, Statistics, and Probability

Process Strands:

Problem solving, Reasoning, and Proof
Communications, Connections, and Representations

Major Concepts:

Number Sense
Efficiency of Number Sense
Mental Mathematics
Relative Magnitude

Proportionality
Ratios
Rates

Percents
Fractions
Similarity

Area and Volume
Scale Factors
Conceptual Understanding of Area and Volume

Equality
Conceptual Understanding of Variables
Congruence

Functions
Patterns
Rates of Change

Conducting Statistical Studies
Read, Interpret, and Analyze Representations and Create Representations
Algebra Through Data
Informal Testing of Hypotheses

Major Stems:

Demonstrates Conceptual Understanding of Rational Numbers

Demonstrates Understanding of the Relative Magnitude of Numbers

Accurately Solves Problems involving Proportional Reasoning

Applies Properties of Numbers and Field Properties

Uses a Variety of Mental Computation Strategies to Solve Problems

Demonstrates Conceptual Understanding of Area and Volume

Identifies, Extends, and Generalizes a Variety of Patterns

Demonstrates Conceptual Understanding of Linear and Non-Linear Functions and Relations

Demonstrates Conceptual Understanding of Algebraic Expressions

Demonstrates Conceptual Understanding of Equality

In Response to a Teacher or Student Generated Question or Hypothesis

Mathematical Investigations and Concepts for Algebraic Reasoning
COURSE CONTENT COMPETENCIES

1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.*
2. *Students will understand arithmetic operations on real numbers, the relationships among those operations, and the relationship among those operations and their properties.*
3. *Students will understand the effect that various operations have on different types of numbers.*
4. *Students will understand that proportional reasoning can be applied to a variety of mathematical contexts.*
5. *Students will understand that spatial reasoning and measurement can be used to represent, describe, and make sense of the world in which they live.*
6. *Students will understand that patterns, relations, models, and functions can be used to describe, interpret, and predict real world phenomena.*
7. *Students will understand that tables, graphs, and equations are ways for depicting and analyzing patterns of change in data.*
8. *Students will understand that variables can be used in various ways and in symbolic statements which can be manipulated by mathematical rules to produce equivalent expressions.*
9. *Students will understand the statistical process.*
10. *Students will understand that probability is a tool for statistics.*
11. *Students will understand the nature of variability in each component of the statistical process.*

Mathematical Investigations and Concepts for Algebraic Reasoning
COURSE PROCESS SKILLS AND HABITS OF THE MIND

(IMPORTANT: These process skills and habits of the mind are integrated throughout the instruction of the major stems and should be present throughout performance assessments.)

1. *Students will understand that a variety of problem-solving strategies can be used to investigate everyday as well as increasingly complex mathematical situations.*
2. *Students will understand that exploring, justifying, and synthesizing mathematical conjectures are part of systemic reasoning which is common to all content areas and a defining feature of mathematics.*
3. *Students will understand that actively exploring, investigating, describing, and explaining mathematical ideas promotes communication which leads to a greater comprehension of mathematical concepts.*
4. *Students will understand that mathematical connections will help them become aware of the usefulness of mathematics, serve to bridge the concrete and the abstract, and enable deeper understanding of important ideas.*
5. *Students will understand that representing ideas and connecting the representations lies at the heart of understanding mathematics.*

6. *[Students will understand that mathematical thinking extends beyond important mathematical ideas and procedures and involves important modes of thought or Habits of Mind (e.g., risk-taking, perseverance, making conjectures, challenging solutions .]*

Mathematical Investigations and Concepts for Algebraic Reasoning

Number and Operations - Stem 1 **(Note: Numbering of the stems follows the number convention in the GSEs. For example, this particular stem is numbered stem 1 since it is the first stem in the Number and Operations strand in the GSE document NOT because it is the first stem listed on page 2 of this document – that is coincidental.)**
Demonstrates Conceptual Understanding of Rational Numbers

Competencies	1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.*
Knowledge/Content	1. *Model and interpret various forms of real numbers.*

Number and Operations - Stem 2
Demonstrates Understanding of the Relative Magnitude of Numbers

Competencies	1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.*
Knowledge/Content	1. *Order, compare, and identify equivalent rational numbers across number formats.*

Number and Operations - Stem 4
Accurately Solves Problems involving Proportional Reasoning

Competencies	1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.* 2. *Students will understand the effect that various operations have*

	on different types of numbers. *4. Students will understand that proportional reasoning can be applied to a variety of mathematical contexts.* *5. Students will understand that patterns, relations, models, and functions can be used to describe, interpret, and predict real world phenomena.* *6. Students will understand that tables, graphs, and equations are ways for depicting and analyzing patterns of change in data.*
Knowledge/Content	*1. Apply proportional reasoning appropriately within mathematical contexts.* *2. Connect proportionality to various multiplicative relationships.* *3. Determine when relationships are directly proportional and when relationships are indirectly proportional.* *4. Distinguish between linear relationships that are directly proportional and those that are not.*

Number and Operations - Stem 6
Uses a Variety of Mental Computation Strategies to Solve Problems

Competencies	*1. Students will understand the real number system, relationships among numbers, and various ways of representing numbers.* *2. Students will understand arithmetic operations on real numbers, the relationships among those operations, and the relationship among those operations and their properties.* *3. Students will understand the effect that various operations have on different types of numbers.*
Knowledge/Content	*1. Uses a variety of mental computation strategies to solve problems (e.g., using compatible numbers, applying properties of operations, using mental imagery, using patterns).* *2. Uses a variety of mental computation strategies to determine the reasonableness of answers.*

Number and Operations - Stem 8
Applies Properties of Numbers and Field Properties

Competencies	*1. Students will understand the real number system, relationships among numbers, and various ways of representing numbers.* *2. Students will understand arithmetic operations on real numbers, the relationships among those operations, and the relationship among those operations and their properties.* *3. Students will understand the effect that various operations have*

	on different types of numbers.
Knowledge/Content	1. *Connect number operations to functions (e.g., finding the whole numbers that multiply to 24 is equivalent to determining the whole numbered ordered pairs of the function y = 24/x).* 2. *Apply Field Properties to become more efficient with mental mathematics.*

Sample Performance Assessment (SPA) for Number and Operations	A new cell phone company wants to attract more customers. The marketing department of the company is considering using the following slogan, "Pay ONLY for the minutes YOU USE." Does this slogan mean the relationship between the number of minutes utilized and cell phone costs is directly proportional? Explain your reasoning. Develop two fee structures for this company based on real world costs. One fee structure should be directly proportional and the other should not. Include graphs and tables with your fee structures. As the mathematics consultant for the company, which fee structure would you recommend the company to implement if they use the proposed slogan? Use mathematical reasoning and understanding to justify your choice.
Mathematics Competencies Addressed in SPA	*4. Students will understand that proportional reasoning can be applied to a variety of mathematical contexts.* *6. Students will understand that patterns, relations, models, and functions can be used to describe, interpret, and predict real world phenomena.* *7. Students will understand that tables, graphs, and equations are ways for depicting and analyzing patterns of change in data.*

SPA Number and Operations			
Level 4	**Level 3**	**Level 2**	**Level 1**

Explanation relating to slogan demonstrates a clear and thorough understanding of directly proportional relationships. Two fee structures are based on real world costs and identify the differences of relationships which are directly proportional and not directly proportional. Accurate graphs and tables are included to support the two fee structures. Fee structure recommendation to company demonstrates mathematical reasoning and justification based on proportional reasoning and an understanding of directly proportional relationships.	Explanation relating to slogan demonstrates some key understandings of directly proportional relationships. Two fee structures are based on real world costs and identify some differences of relationships which are directly proportional and not directly proportional. Graphs and tables are included to support the two fee structures but have minor errors in either the graph or table. Fee structure recommendation to company demonstrates mathematical reasoning and justification based on some understanding of directly proportional relationships.	Explanation relating to slogan demonstrates only one or two key understandings of directly proportional relationships. Two fee structures are based on real world costs but do not identify the differences of relationships which are directly proportional and not directly proportional. Graphs and tables are included to support the two fee structures but inconsistencies exist in both. Provides a fee structure recommendation to the company without mathematical reasoning and justification.	Explanation relating to slogan is not provided or explanation is not relevant. One or two fee structure(s) are provided but not based on real world costs. Differences of relationships which are directly proportional and not directly proportional are not attempted. A graph or table is included which may or may not support the given fee structure(s). No fee structure recommendation is provided or the recommendation is not relevant to the task.

Sample Performance Assessment (SPA) for Number and Operations	Your local school board has decided to focus on the how mathematics should be taught in classes at your school. Half of the members on the school board think mathematical algorithms should be mastered by all students in your school. The other half believes mental mathematics should be emphasized. Develop a survey which determines the various strategies everyday people use to perform mathematical calculations. In order to develop your survey, you will need to conduct research relating to mathematical algorithms, number sense and mental mathematics. In addition you need to research various strategies used to perform mathematical calculations. Once you have developed your survey, determine how you will sample the population to gather information based on your survey. Conduct your sampling plan. After you have gathered information based on your survey, develop a presentation and report for your local school board. Remember, your school board is split on their views regarding algorithms and mental mathematics. Your role as a presenter is to provide your judgment regarding this debate and to provide research and evidence to support your judgment. Personalize your presentation to include your own strengths and growth areas relating to mathematical algorithms and mental mathematics.

Mathematics Competencies Addressed in SPA	1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.* 2. *Students will understand arithmetic operations on real numbers, the relationships among those operations, and the relationship among those operations and their properties.* 3. *Students will understand the effect that various operations have on different types of numbers.* 9. *Students will understand the statistical process.*

SPA Number and Operations			
Level 4	**Level 3**	**Level 2**	**Level 1**
Presents research and evidence which provides a thorough understanding of mathematical algorithms, number sense and mental mathematics. Demonstrates and provides various strategies (more than five) for performing mathematical calculations. Develops a survey with strong connections to identified research and conducts a sampling plan which adequately represents a targeted population. Provides a presentation and report that encompasses all sides of the debate comparing mathematical algorithms to mental mathematics. Presentation includes a strong self assessment of personal strengths and growth areas relating to mathematical algorithms and mental mathematics.	Presents research and evidence which provides a basic understanding of mathematical algorithms, number sense and mental mathematics. Demonstrates and provides various strategies (in the range of 3 to 5 strategies) for performing mathematical calculations. Develops a survey with some connections to identified research and conducts a sampling plan with a limited target population. Provides a presentation and report including some of the issues relating to the debate comparing mathematical algorithms to mental mathematics. Presentation includes a basic self assessment of personal strengths and growth areas relating to mathematical algorithms and mental mathematics.	Presents minimal research and evidence relating to mathematical algorithms, number sense and mental mathematics. Demonstrates and provides two to three different strategies for performing mathematical calculations. Develops a survey with minimal connections to identified research and conducts a sampling plan with only one subgroup of a specific population (e.g., surveys only students in a specific grade level). Provides a presentation and report including one or two issues relating to the debate comparing mathematical algorithms to mental mathematics. Presentation includes a minimal self assessment of personal	Presents very little research and evidence (or may be irrelevant) relating to mathematical algorithms, number sense and mental mathematics. Demonstrates and provides only one or two different strategies for performing mathematical calculations. Develops a survey with little to no connections to identified research and conducts a sampling plan without defining a target population. Provides a presentation and report which may contain irrelevant issues relating to the debate comparing mathematical

		strengths and growth areas relating to mathematical algorithms and mental mathematics.	algorithms to mental mathematics. Presentation does not include a self assessment of personal strengths and growth areas relating to mathematical algorithms and mental mathematics.

Mathematical Investigations and Concepts for Algebraic Reasoning

Geometry and Measurement - Stem 6
Demonstrates Conceptual Understanding of Area and Volume

Competencies	*4. Students will understand that spatial reasoning and measurement can be used to represent, describe, and make sense of the world in which they live.*
Knowledge/Content	*1. Apply the concepts of area and volume to real world situations and various applications (e.g., Pick's Theorem).* *2. Apply scale factors and determine their effect on the linear dimensions of polygon and on area, volume, and surface area of three-dimensional figures.*

Sample Performance Assessment for Geometry and Measurement	**How many different polygons of area 2 square units can be made on a standard 25-pin geoboard? Different means that the polygons are non-congruent.** Over the next several days, you will research this open-ended mathematics problem that involves area, congruence, problem solving, reasoning, and communication (just to name a few!). Although you will work in groups in class, you will need to spend time on the problem each night. The goal is not necessarily to arrive at a single right answer, but to practice persistence in researching a problem in mathematics. Unlike

	normal homework, this investigation may take more than a half hour. We strongly suggest that you explore different ideas and strategies to find solutions to the problem. After every work session, you should make a journal entry discussing what you did, what you learned, breakthroughs, new strategies, etc. Additionally, you will summarize your work in a Statement of Reflection at the end of the project. Good luck, have fun, and work hard!!
Mathematics Competencies Addressed in SPA # 1	1. *Students will understand that spatial reasoning and measurement can be used to represent, describe, and make sense of the world in which they live.*

SPA Geometry & Measurement		
Points Earned	**Possible Points**	**Description**
	16	Determined based on number of polygons found
	4	No duplicate polygons
	4	No polygons with incorrect area
	4	Journal (with entries from every work session)
	4	Statement of reflection on your work and experience (see below – 1 to 1 ½ pages double spaced and typed)
	2	Your scoring key (self scored and totaled)
	4	Individual work compared to group work
	4	On task, behavior, effort, and attitude
	8	Overall presentation of final copy (each polygon should be labeled with a number, each page labeled with page number and number of polygons, neatness, quality, etc…)

Total	50	

Statement of Refection
(include the following components)

- What went well?
- What went wrong?
- What methods did you try and/or use?
- What possible extensions to the problem can you think of?
- If you did it again, what would ***you*** do differently?
- How hard did you work?
- How long did you work?
- How well did you work with others?
- How did your group interact?
- What did you like best about the project?
- What did you like the least about the project?
- What was the most interesting thing you learned?
- What was the most important thing you learned?
- If I assigned the project again, what should ***I*** change or do differently?
- **Explain all of the above – why or why not!**

Mathematical Investigations and Concepts for Algebraic Reasoning

Functions and Algebra - Stem 1
Identifies, Extends, and Generalizes a Variety of Patterns

Competencies	5. *Students will understand that patterns, relations, models, and functions can be used to describe, interpret, and predict real world phenomena.* 6. *Students will understand that tables, graphs, and equations are ways for depicting and analyzing patterns of change in data.* 7. *Students will understand that variables can be used in various ways and in symbolic statements which can be manipulated by mathematical rules to produce equivalent expressions.*
Knowledge/Content	1. *Use a variety of patterns to interpret, represent, and model real world phenomena (e.g., using motion detectors to collect data and determining functions that model the data).*

Functions and Algebra - Stem 2
Demonstrates Conceptual Understanding of Linear and Non-Linear Functions and Relations

Competencies	1. *Students will understand the real number system, relationships among numbers, and various ways of representing numbers.* 2. *Students will understand arithmetic operations on real numbers, the relationships among those operations, and the relationship among those operations and their properties.* 3. *Students will understand the effect that various operations have on different types of numbers.* 4. *Students will understand that patterns, relations, models, and functions can be used to describe, interpret, and predict real world phenomena.* 5. *Students will understand that tables, graphs, and equations are ways for depicting and analyzing patterns of change in data.* 6. *Students will understand that symbolic statements can be manipulated by mathematical rules to produce equivalent statements.*
Knowledge/Content	1. *Connect number operations to functions (e.g., addition number facts form linear relationships, multiplication number facts form exponential relationships).* 2. *Apply and interpret rates of change in real world contexts through the use of formulas, graphs, and verbal representations (e.g., translating from stories to graphs and from graphs to stories).*

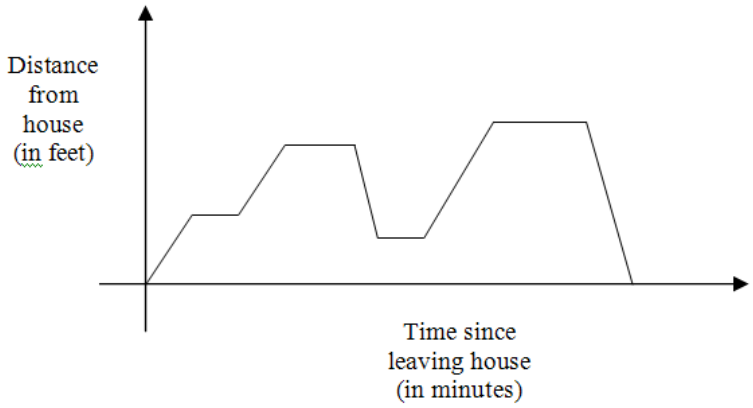
Functions and Algebra - Stem 3
Demonstrates Conceptual Understanding of Algebraic Expressions

Competencies	7. *Students will understand that variables can be used in various ways and in symbolic statements which can be manipulated by mathematical rules to produce equivalent expressions.*
Knowledge/Content	1. *Provide examples of situations where variables are used as unknowns, in generalizations, as quantities that vary, and in formulas.* 2. *Connect the various ways that variables are used to the various ways that the equal sign is used in mathematics.*

Functions and Algebra - Stem 4
Demonstrates Conceptual Understanding of Equality

Competencies	7. *Students will understand that variables can be used in various ways and in symbolic statements which can be manipulated by mathematical rules to produce equivalent expressions.*
Knowledge/Content	1. *Model equality as a relation and provide examples of situations where the equal sign is used for conditional equality, identically equivalent expressions, and definitional equality.* 2. *Solve problems using algebraic reasoning about equality (e.g., see item #48 in the GSE Support Materials, www.ed.state.nh.us/education/math.)*

Sample Performance Assessment (SPA) for Functions and Algebra	1. For the following real world scenario: a.) sketch a complete graph showing how the distance from home is related to time; and b.) thoroughly explain the reasoning behind your sketch including all parts of the graph. An elementary student began his walk to school at a steady rate. After several minutes he realized he had forgotten his homework and ran back home. After spending some time searching for it, he began to head to school again faster than his first attempt, but steady just the same. Soon after passing the point where he headed back to get his homework, he slowed down a bit. Suddenly, he heard some of his friends coming, so he ducked behind a tree and waited for them to pass. As they passed, he jumped out and scared them. They all had a good laugh and continued on their way skipping, then walking, then running. Thankfully, the student's mom happened to come by and gave them a ride the rest of the way to school. 2. Write a short story that could describe the given graph that represents a child's trip to school.

	 3. Additionally, come up with your own story. Provide an answer key that includes parts (a) and (b) as in Question 1.

SPA Functions & Algebra			
Level 4	**Level 3**	**Level 2**	**Level 1**
In task 1, the student sketches a correct graph for the given scenarios and explains all significant portions of the graph. In task 2, the student writes a story that describes the given graph taking into account all portions of the graph. In task 3, the student comes up with a story and graph at the level described above with many different rates such as varying positive and negative slopes and constant and vertical slopes. All axis are labeled correctly.	The student has no significant errors or omissions (same as Level 4 but with minor errors. In task 3, the student's story and graph contains multiple rates of change. Overall, no more than one or two "minor" mistakes are made (i.e. wrong labeling or incomplete reasoning).	One of the above answers to the three parts are incorrect or missing; or, two significant mistakes in two of the 3 answers.	Two of the above answers to the three parts are incorrect or missing; or three or more significant mistakes.
Sample Formative Assessment for Functions and Algebra (equality)	Designing questions to help elicit students' understandings of equality can be an informative exercise. Furthermore, research indicates that students clarify their notations of equality when their existing conceptions are challenged. The following formative assessment questions elicit four benchmarks of equality that researchers have identified which can serve as a framework for designing questions that intentionally challenge students' conceptions of equality – developing an understanding of the equal sign in a relational sense, developing an		

	understanding of alternative formats involving the equal sign, developing an understanding of the equal sign in a relational sense, developing an understanding of the equal sign in a relational sense and use relational thinking (Carpenter et al., 2003 as described in Henderson & Christensen, 2006). The purpose of giving this task is to assess where students are in relation to the four benchmarks of equality. Work through the following ten questions. In questions 1–3, fill in the box with the correct number. Explain how you know that the number you put in the box is correct. 1) 8 + 5 = □ + 2 2) 47 + 21 = 45 + □ 3) 8 × 3 = 4 × □ For question 4, fill in the box and the circle with the correct number. Explain how you know that the number you put in the box and the number that you put in the circle is correct. 4) 22 + 18 = □ + 12 = ○ In questions 5–10, determine if each statement is true or false. Explain how you know. 5) 8 = 5 + 3 6) 5 + □ = □ + 5 7) 24 + 13 = 21 + 16 8) 14 × 8 = 7 × 16 9) 12 × 46 = 24 × 23 10) 3 + 7 = 10 + 5 = 15 + 2 = 17 (From Andrusiak, R. (2007). *Understanding Equality – A Researched-Based Unit of Study for Elementary School Teachers*. Rhode Island Department of Education.)

Mathematical Investigations and Concepts for Algebraic Reasoning

Data, Statistics, and Probability - Stem 6
In Response to a Teacher or Student Generated Question or Hypothesis

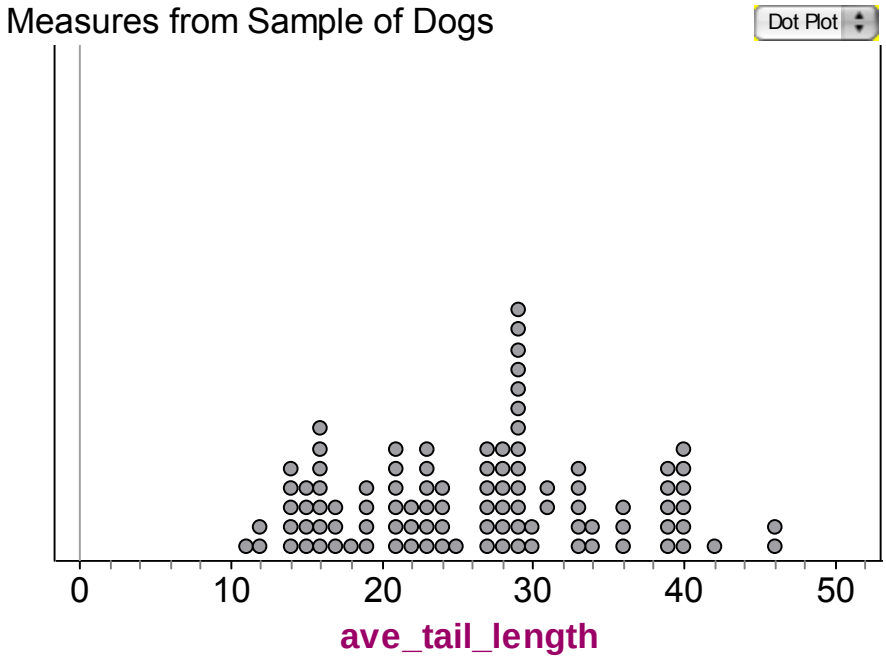
Competencies	1. *Students will understand the statistical process.* 2. *Students will understand that probability is a tool for statistics.*

	3. *Students will understand the nature of variability in each component of the statistical process.*
Knowledge/Content	1. *Formulate questions, collect data, analyze data, and interpret the results.* 2. *Use investigations and simulations to understand that probability is a tool for statistics.* 3. *Anticipate variability when formulating questions, design data collection methods, using distributions to analyze data, and interpreting results.*

Sample Performance Assessment for Data, Statistics, and Probability	Quinn's dog day care has been accused of rejecting dogs with long tails. The most recent data show that out of ten dogs considered for the day care service, two were rejected. The two dogs rejected had an average tail length of 42 cm. The tail lengths of the ten dogs that were being considered for the day care are: 12 cm, 10 cm, 14 cm, 24 cm, 44 cm, 48 cm, 36 cm, 18 cm, 22 cm, 34 cm 1) Do you think that the dog day care service is discriminating against dogs with long tails? Explain. 2) Design a simulation to determine how likely it is by chance alone to select two dogs for rejection whose average tail length is at least 42 cm. Display the sampling distribution of the sample averages. 3) Based on your results from number 2, determine if the day care center has some explaining to do. Support your conclusions. (Adapted from Watkins, A.E., Scheaffer, R.L., & Cobb, G.W. (2004). *Statistics in Action – Understanding a World of Data*. Emeryville, CA: Key Curriculum Press.)

SPA Data, Statistics, & Probability			
Level 4	**Level 3**	**Level 2**	**Level 1**

A hypothesis is given to task 1 with appropriate support, which may include reasoning about how many ways you can obtain two dogs whose average tail length is at least 42 cm. A simulation is designed (by hand or using appropriate technology) in task 2 to determine the likelihood of obtaining an average tail length of 42 cm of more and includes a clear explanation, in task 3, of what the sampling distribution displays (e.g., Out of 100 random samples of two dogs, only 3 times was the average tail length 42 cm or more, so it is unlikely that by chance alone we would obtain this average or larger. So, I believe that the day care center has some explaining to do.). Explanation may include the theoretical probability using counting techniques. A Sample Distribution is given below the rubric.	A hypothesis is given to task 1 with appropriate support, which may include reasoning about how many ways you can obtain two dogs whose average tail length is at least 42 cm. A simulation is designed (by hand or using appropriate technology) in task 2 to determine the likelihood of obtaining an average tail length of 42 cm of more and includes an explanation, in task 3, of what the sampling distribution displays but may contain minor errors.	A hypothesis is given to task 1 with appropriate support, which may include reasoning about how many ways you can obtain two dogs whose average tail length is at least 42 cm. A simulation is designed (by hand or using appropriate technology) in task 2 to determine the likelihood of obtaining an average tail length of 42 cm of more, but the simulation not be carried out appropriately or the explanation of what the sampling distribution shows isn't clear or an inappropriate conclusion is made from the results.	A hypothesis is given to task 1 but may lack support. The simulation may not be carried out appropriately and the conclusion is inappropriate, or the task is incomplete.



Appendix A
Resources

Printed Resources

American Association for the Advancement of Science. (1993). Benchmarks for Science Literacy. New York, NY: Oxford University Press, Inc.

Batanero, C., & Sanchez, E. (2005). What is the Nature of High School Students' Conceptions and Misconceptions about Probability? In Jones, G.A. (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning*. New York, NY: Springer.

Billings, Ester M.H. (2002). Cocoa: Teaching Notes. In B. Litwiller & G. Bright (Eds.), *Classroom Activities for Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (chpt. 21). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Bright, G.W., Brewer, W., McClain, K., & Mooney, E.S. (2003). *Navigating Through Data Analysis in Grades 6–8*. Friel, S.N.., & House, P.A. (Eds.). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Bright, G.W., Frierson Jr., D.F., Tarr, J.E., & Thomas, C. (2003). *Navigating Through Probability in Grades 6–8*. Friel, S.N.., & House, P.A. (Eds.). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Cai, J. & Sun, W. (2002). Developing Students' Proportional Reasoning: A Chinese Perspective. In B. Litwiller & G. Bright (Eds.), *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook* (chpt. 21). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Carpenter, T.P., Franke, M.L., & Levi, L. (2003). *Thinking Mathematically – Integrating Arithmetic & Algebra in Elementary School*. Portsmouth, NH: Heinemann.

Cramer, K. & Post, T. (1993). *Connecting Research to Teaching Proportional Reasoning*. Mathematics Teacher, 86(5), 404-407.

Cramer, K., Post, T. & Currier, S. (1993). Learning and Teaching Ratio and Proportion: Research Implications. In Owens, D.T. (Ed.), *Research Ideas for the Classroom – Middle Grades Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Donovan, M.S., & Bransford, J.D. (Eds.). (2005). *How Students Learn – Mathematics in the Classroom*. Washington, D.C.: National Academies Press.

Jones, G.A. (2005). *Exploring Probability in School: Challenges for Teaching and Learning*. New York, NY: Springer.

Jones, G.A., & Thornton, C.A. (2005). An Overview of Research into the Teaching and Learning of Probability. In Jones, G.A. (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning*. New York, NY: Springer.

Keeley, Page., & Rose, Cheryl. (2005). *Mathematics Curriculum Topic Study*. Thousand Oaks, CA: Corwin Press.

Kieran, C. (1999). *The Learning and Teaching of School Algebra*. In B. Moses (ED). *Algebraic Thinking, Grades K-12 – Readings from NCTM's School-Based Journals and Other Publications*. National Council of Teachers of Mathematics. Reston, VA. (Original work published in 1992)

Kieran, C., & Chalouh, L. (1993). Prealgebra: The Transition from Arithmetic to Algebra. In Owens, D.T. (Ed.), *Research Ideas for the Classroom – Middle Grades Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Kilpatrick, J., Martin, W.G., & Schifter, D. (Eds.). (2003). *A Research Companion to Principles and Standards for School Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

Lamon, S.J. (1999). *Teaching Fractions and Ratios for Understanding*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.

Litwiller, B. & Bright, G. (Eds.). (2002). *Making Sense of Fractions, Ratios, and Proportions: 2002 Yearbook*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*. Reston, VA: Author.

National Research Council. (2001). *Adding it up: Helping Children Learn Mathematics*. Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Rose, Cheryl. (2007). *Uncovering Student Thinking in Mathematics: 25 Formative Assessment Probes.* Thousand Oaks, CA: Corwin Press.

Shaughnessy, J.M., Barrett, G., Billstein, R., Kranendonk, H.A., & Peck, R. (2004). *Navigating Through Probability in Grades 9–12*. Lott, J.W., & House, P.A. (Eds.). Reston, VA: The National Council of Teachers of Mathematics, Inc.

Shaughnessy, J.M., & Bergman, B. (1993). Thinking about Uncertainty: Probability and Statistics. In Wilson, S.P. (Ed.), *Research Ideas for the Classroom – High School Mathematics*. Reston, VA: The National Council of Teachers of Mathematics, Inc.

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On-line Resources

GeoGebra, http://www.geogebra.org/cms/.

Guidelines for Assessment and Instruction in Statistical Education (GAISE), http://www.amstat.org/education/gaise/.

Illuminations, http://illuminations.nctm.org/.

National Library of Virtual Manipulatives, www.nlvm.usu.edu

Numb3rs Activities, http://www.weallusematheveryday.com/tools/waumed/home.htm.

PBS Teacher Source, http://www.pbs.org/teachers/.

Texas Instruments Activity Exchange, http://education.ti.com/educationportal/activityexchange/activity_list.do?cid=us.

Toolkit for Change, www.toolkitforchange.org.

Appendix B
Recommended Distribution of Emphasis

This emphasis does not translate directly to instructional time (e.g., some topics in statistics may take longer to develop than topics in number and operations). However, the emphasis represents the percent of score points in each of the strands that would be recommended for a final project, final exam, capstone project, or other means of final assessment. Process standards should be integrated throughout the content strands.

(IMPORTANT: The process skills and habits of the mind are integrated throughout the instruction of the major stems and should be present throughout performance assessments.)

Numbers & Operation	Geometry & Measurement	Functions & Algebra	Data, Statistics, & Probability
30%	20%	30%	20%

Appendix C
Habits of the Mind

HABITS of MIND

We would like students to:

1. Develop a "curiosity" about mathematics and seek the "whys" behind the "how to's";
2. Recognize that hard work, persistence, and risk-taking are needed to do mathematics;
3. Develop and demonstrate self-confidence when doing mathematics and analytical thinking;
4. Value the process of exploration and investigation of mathematical concepts;
5. Make and test conjectures and verify or contradict those conjectures;
6. Recognize the need for logical arguments, proofs and deductive reasoning to verify conjectures and recognize that intuitive explanations are important;
7. Realize the need to communicate mathematics in writing and orally;
8. Recognize the role of estimation and the need to examine the reasonableness of results;
9. Recognize the need to employ strategies and heuristics in solving problems and when to employ them;
10. Demonstrate that solving problems in mathematics involves analyzing examples and appreciating the subtleties of an assumption or its limitations;
11. Value the use of technology, learning to apply it only when needed and appropriate, and recognize that technology does not replace knowledge of basic facts and skills;
12. Appreciate that mathematics is the language of nature and science and is a tool for quantitative reasoning;
13. Recognize that failure is a fact of life and that to be successful at challenges, one will experience failure, but will, hopefully, learn from it; and
14. Work as a member of a group, but proceed independently to draw inferences.

"Acquisition of these habits of mind supercedes mastery of content knowledge. Their achievement is necessary because It is not enough to know something; the learner must possess the ability to do something with that knowledge, whether it is to solve a problem, reach a conclusion or present a point of view."

(Bob Karsky, Report Summary: Understanding University Success)

The "Habits of Mind" slides are adapted and compiled from Understanding University Success and A Consensus Model for Preservice Teacher Education in Mathematics and Science.